expressions for the characteristic impedances presented in this paper are believed to be accurate enough for practical purposes, are all in closed form, and are easily manipulated. Wheeler [8] gives a more accurate and useful analysis, but his results are more complicated. Other good work (such as [9]) exists, but their results are also more complicated. The attenuation constant formula given here is only approximately correct, and may be of value for a quick estimation, and it happens often that experimental values of the attenuation constant show a marked deviation from theoretical values!

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Impedance of an Elliptic Conductor Arbitrarily Located Between Ground Planes Filled with Two Dielectric Media

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Abstract —**This paper presents a method of determining the characteristic impedance of an ellipse arbitrarily located between parallel conducting planes when the region between the planes is filled with two different dielectric media. The same generalized formulation is then extended to the case when one of the ground planes is moved to infinity. The impedance data for various locations of the dielectric interface with respect to the conductor of elliptic and circular cross sections are presented. The results of some of the special cases are compared with those available in the literature [2].**

I. Introduction

The method of analysis of transmission-line properties of parallel strips separated by a dielectric was carried out by Wheeler in 1965 [1]. He has also analyzed the properties of a round wire in a cylindrical shield of polygonal cross section [2]. Later studies were made on an offset stripline and microstripline using planar strip when the line is filled with two dielectric media [3]. The analysis of transmission line for the case when the center conductor assumes the form of an ellipse arbitrarily located between ground planes or placed above a single ground plane has been carried out by the authors [4].

In the present work, the generalized conformal transformation obtained for the conductor with an elliptic boundary is used for the estimation of impedance data when the region between the ground planes of the line is filled with two different dielectric media. This analysis is based on the quasi-TEM-mode approximation and, hence, is valid at the lower frequency ranges only. The conformal transformation transforms one half of the structure into a parallel-plate configuration in which the top and bottom plates correspond, respectively, to the conductor with the curved boundary and the ground planes. In this transformed parallel-plate configuration, the dielectric interface appears in the form of a curved contour. The characteristic impedance of this composite parallel-plate structure is calculated by considering the series parallel combinations of small incremental capacitances [1]. The expressions for these capacitances appear in the form of integrals which are numerically evaluated using the adaptive quadrature method [5].

The formulation is used for the computation of characteristic impedances for the cases of a center conductor 1) placed exactly above the dielectric interface, 2) placed such that one of the principal axes is coplanar with the dielectric interface, and 3) fully embedded in the dielectric.

II. Formulation for the Case of an Ellipse Embedded in Mixed Dielectrics Between Ground Planes

Consider the configuration shown in Fig. 1(a). The method of transforming the half *FABCDEF* into a parallel-plate configuration as shown in Fig. 1(b) has already been developed by the authors [4]. The transformation establishes the relation between the points in the W-plane (Fig. 1(b)) with those of the corresponding points in the Z-plane (Fig. 1(a)). The corresponding loci in the two planes (Z and W planes) can also be determined. The equations for the lines parallel and perpendicular to the two ground planes in the W-plane are given by [6]

$$u' = -\frac{1}{K(m)}\left[F(\beta|m) + F\left(\sin^{-1}\sqrt{\frac{n}{m}}\middle|m\right)\right] \tag{1}$$

$$\frac{v'}{V_0} = 1 - \frac{F(\gamma|m_1)}{K'(m)} \tag{2}$$

where m, n are constants, $0 < n < m < 1$, and where F and K correspond, respectively, to incomplete and complete elliptic integrals of the first kind with given argument and modulus. β and γ are real arguments of the elliptic integrals. The elliptic integral with complex argument Φ can be expressed as

$$F(\Phi|m) = F(\eta + i\xi|m) = F(\beta|m) \pm iF(\gamma|m_1)$$

$$t_r = \cosh\xi \sin\eta = \frac{\sin\beta\sqrt{1 - m_1\sin^2\gamma}}{\cos^2\gamma + m\sin^2\beta\sin^2\gamma}$$

$$t_i = \cosh\eta \sinh\xi = \frac{\cos\beta\cos\gamma\sin\gamma\sqrt{1 - m\sin^2\beta}}{\cos^2\gamma + m\sin^2\beta\sin^2\gamma}$$

$$t = t_r + it_i = \sin\Phi, \qquad m_1 = (1 - m).$$

U_0 and V_0 shown in Fig. 1(b) are given by

$$U_0 = 1 + \frac{F\left(\sin^{-1}\sqrt{n/m}\,|m\right)}{K(m)} \tag{3}$$

$$V_0 = \frac{K'(m)}{K(m)}. \tag{4}$$

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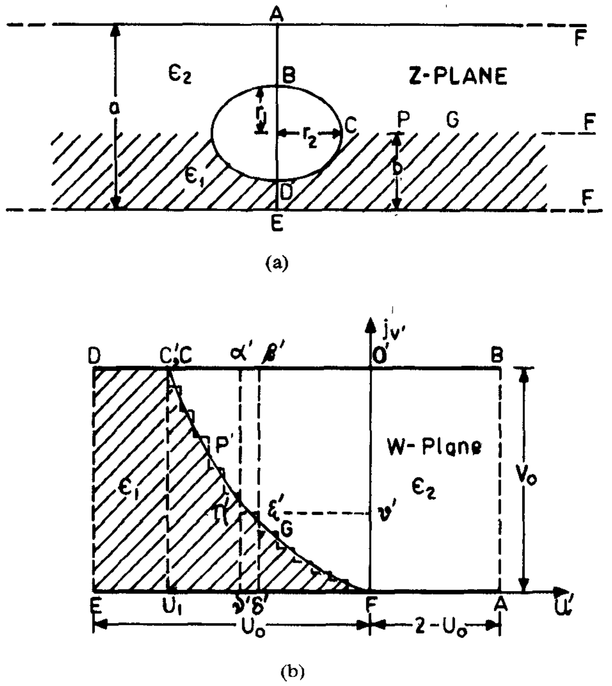


(a)

(b)

Fig. 1. Conformal representation of a transmission line with an elliptic center conductor between two infinite parallel planes filled with two-layered dielectric media.

For any point P in the Z-plane, its transformation P' is an intersection of a particular combination of $u'=$ constant and $v'=$ constant lines. The variables β and γ, which appear in (1) and (2), are not obtained as an explicit function of variable Z. On the other hand, the expression for Z is obtained as the explicit function of the variables β and γ. The transformation of $u'=$ constant and, hence, $\beta=$ constant contour is in the form of a curve in the Z-plane. Different points on the curve correspond to different values of v' and, hence, γ. If a particular locus in the Z-plane is specified, the transformed set of discrete points on the locus lies on a $u'=$ constant and, hence, a $\beta=$ constant contour. It is, therefore, possible to start with a particular value of β and vary γ over the range 0 to $+\pi/2$ until the desired point on the Z-plane is obtained using the expressions available in the literature [4]. From a knowledge of values of β and γ, the transformation of a set of discrete points in the W-plane are obtained.

In the problem under investigation, it is necessary to find the transformation of the dielectric interface shown in Fig. 1(a) in the W-plane (Fig. 1(b)). Following the procedure mentioned above, it is found that the dielectric interface appears in the form of a curved contour $C'GF$ as shown in Fig. 1(b). When the lower most and upper most points of the ellipse are on the dielectric interface, point C' in Fig. 1(b) coincides with D and B, respectively. On the other hand, if the dielectric interface coincides with the principal axis, point C' coincides with C, the location of which again depends upon the offset of the center conductor. Since, the dielectric media on the two sides of the line are assumed to be different, the region to the left of the contour $C'GF$ is filled with one dielectric and the region to its right is filled with another. The conformal mapping of the dielectric is valid because it retains the angles of refraction at the interface. The capacitance of the region $C'U_1FO'$ can be estimated by approximating the continuous curve $C'GF$ by an infinitely large number of small steps as shown in Fig. 1(b). The capacitance ΔC_i of the elemental region $\alpha'\beta'\xi'\delta'\nu'\eta'\alpha'$ (Fig. 1(b)) can then be calculated by considering it to be a series combination of two capacitances ΔC_a and ΔC_b of the regions $\alpha'\beta'\xi'\eta'$ and $\eta'\xi'\delta'\nu'$, respectively.

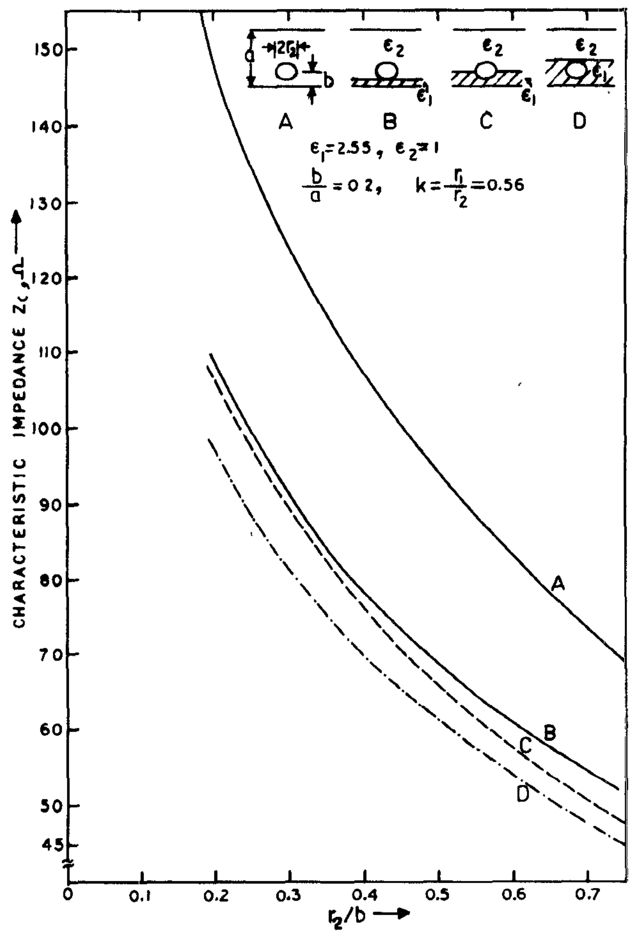


Fig. 2. The variation of the characteristic impedance of a line consisting of an offset ellipse oriented with its major axis parallel to the ground planes as a function of r_2/b with dielectric height as a parameter.

From Fig. 1(b), the values of ΔC_a and ΔC_b are found to be [3]

$$\Delta C_a = \frac{\epsilon_0 \epsilon_2 \, du'}{(V_0 - v')}$$

$$\Delta C_b = \frac{\epsilon_0 \epsilon_1 \, du'}{v'}$$

where V_0 is the separation between the parallel plates of the capacitor (Fig. 1(b)). Using the relation $1/\Delta C_i = 1/\Delta C_a + 1/\Delta C_b$

$$\Delta C_i = \frac{\epsilon_0 \epsilon_1 \epsilon_2 \, du'}{V_0 \left[\epsilon_1 + \frac{v'}{V_0}(\epsilon_2 - \epsilon_1) \right]}. \tag{5}$$

The desired capacitance can be obtained by considering point U_1 as the foot of a perpendicular from point C' on the line EF of Fig. 1(b). The total capacitance of the structure can be evaluated by considering the parallel combination of the three capacitances of the regions $C'DEU_1$, $C'U_1FO'$, and $O'FAB$ of Fig. 1(b).

The capacitance of the region $C'DEU_1$ is given by

$$C_1 = \frac{\epsilon_0 \epsilon_1 (U_0 - U_1)}{V_0}. \tag{6}$$

From (5), the capacitance of the region $C'U_1FO'$ is obtained as

$$C_2 = \int_0^{U_1} \frac{\epsilon_0 \epsilon_1 \epsilon_2 \, du'}{V_0 \left[\epsilon_1 + \frac{f(u')}{V_0}(\epsilon_2 - \epsilon_1) \right]} \tag{7}$$

where the function $f(u')$ represents the curved line $C'GF$.

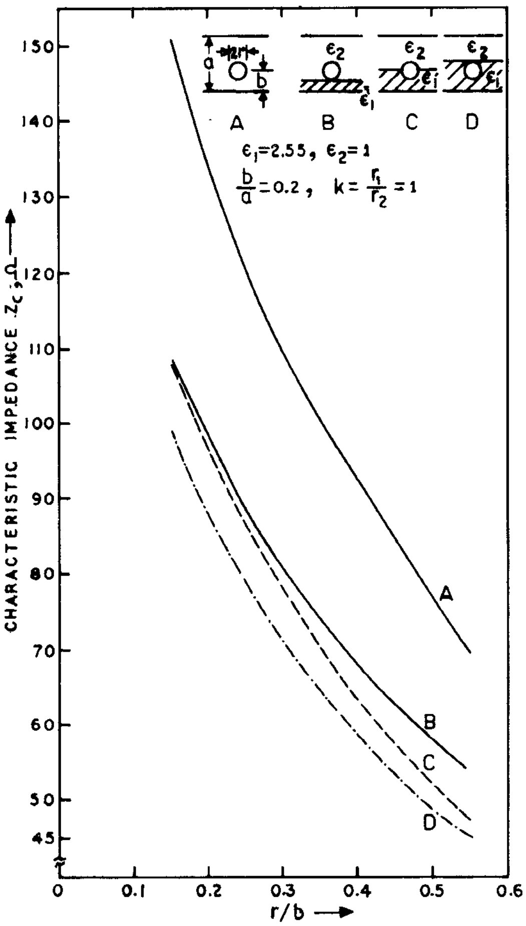


Fig. 3. The variation of characteristic impedance of a line consisting of an offset circular conductor between the ground planes as a function of r/b with dielectric height as a parameter.

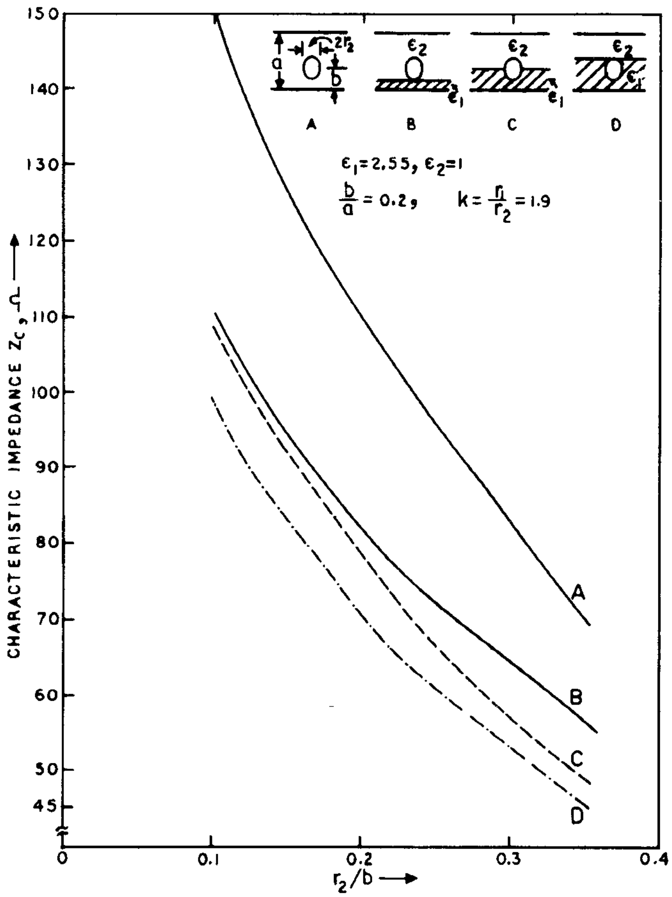


Fig. 4. The variation of the characteristic impedance of a line consisting of an offset ellipse oriented with its major axis perpendicular to the ground planes as a function of r_2/b with dielectric height as a parameter.

The capacitance of the region $0'FAB$ is given by

$$C_3 = \frac{(2-U_0)\epsilon_0\epsilon_2}{V_0}. \quad (8)$$

The total capacitance per unit length of the structure is obtained from the relation

$$C_T = 2(C_1 + C_2 + C_3). \quad (9)$$

The characteristic impedance of the structure is given by [3]

$$Z_c = \sqrt{(C_0/C_T)} \cdot Z_0 \quad (10)$$

where C_0 and Z_0 are, respectively, the capacitance and the characteristic impedance of the structure with air as dielectric. The expression for Z_0 is given by [6]

$$Z_0 = 30\pi \frac{K'(m)}{K(m)}. \quad (11)$$

In order to evaluate the capacitance and the characteristic impedance, it is essential to determine the function $f(u')$ corresponding to a dielectric interface. Determination of $f(u')$ as a function of u' involves use of (1)–(4) of the present paper and [4, eqs. (2)–(5)], which describe the relation between complex variables Z and t developed by the authors [4]. Using this relation between Z and t and (5)–(11) of the present paper, the variation of the characteristic impedance as a function of r_2/b is calculated for $\epsilon_1 = 2.55$, $\epsilon_2 = 1$, and $b/a = 0.2$ for the following cases.

Case 1: The ellipse is so placed that its major axis is parallel to the ground planes and

a) the dielectric interface is tangential to the ellipse at the lowermost point of the minor axis,
b) the major axis coincides with the dielectric interface, and
c) the dielectric interface is tangential to the ellipse at the topmost point of the minor axis.

The numerical results on characteristic impedance for the above configurations with $k = r_1/r_2 = 0.56$ are presented in Fig. 2.

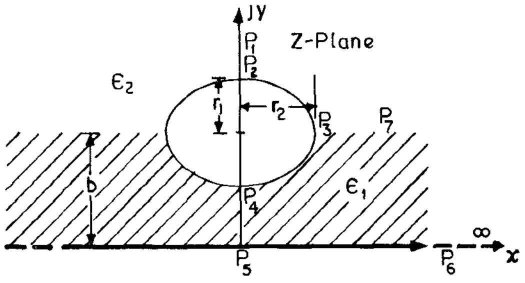
Case 2: The circle is so placed on the dielectric that

d) the lowermost point is on the dielectric interface,
e) the diameter parallel to the ground planes coincides with the dielectric interface, and
f) the topmost point is on the dielectric interface.

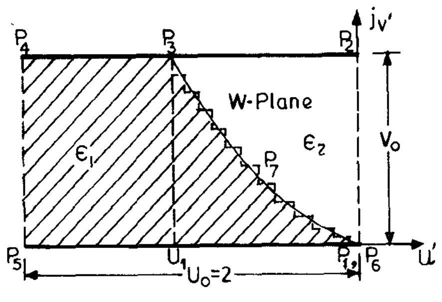
The numerical results on the characteristic impedance for the above configurations with $k = r_1/r_2 = 1$ are presented in Fig. 3.

Case 3: The ellipse is so placed that its minor axis is parallel to the ground planes and

g) the dielectric interface is tangential to the ellipse at the lowermost point of the major axis,
h) the minor axis coincides with the dielectric interface, and
i) the dielectric interface is tangential to the ellipse at the topmost point of the major axis.



(a)



(b)

Fig. 5. Conformal representation of a structure consisting of an ellipse above a ground plane filled with two dielectric media.

The numerical data on impedance for the above configurations with $k = r_1/r_2 = 1.9$ are presented in Fig. 4.

III. Formulation for the Case of an Ellipse Embedded in a Mixed Dielectric Media Placed above a Ground Plane

The relations between the points in the W-plane with the corresponding points in the Z-plane when the top ground plane of Fig. 1(a) is moved to infinity have been developed by the authors [4]. The resulting structure consists of an ellipse above a ground plane as shown in Fig. 5(a). The transformed parallel-plate configuration together with the transformed dielectric interface ($P_3P_7P_6$) is shown in Fig. 5(b). The characteristic impedance for this structure with mixed dielectric media is evaluated following the method similar to that discussed in Section II. Since, in Fig. 5(b), U_0, the distance of the lowermost point of the transformed dielectric interface from the extremity P_5, is equal to 2, the point P_6 coincides with the other extremity of the same conductor in the parallel-plate configuration.

The total capacitance of the structure is therefore evaluated by considering the parallel combination of the capacitances of the regions $P_3P_4P_5U_1$ and $P_2P_1U_1P_3$. The capacitance of the region $P_3P_4P_5U_1$ is given by

$$C_1' = \frac{\epsilon_0\epsilon_1(2-U_1)}{V_0}. \tag{12}$$

The capacitance of the region $P_3U_1P_1P_2$ is obtained as

$$C_2' = \int_0^{U_1} \frac{\epsilon_0\epsilon_1\, du'}{V_0[\epsilon_1 + f_1(u')(1-\epsilon_1)/V_0]} \tag{13}$$

where the function $f_1(u')$ represents the contour $P_1P_7P_3$.

The total capacitance per unit length of the structure is given by

$$C_T' = 2(C_1' + C_2'). \tag{14}$$

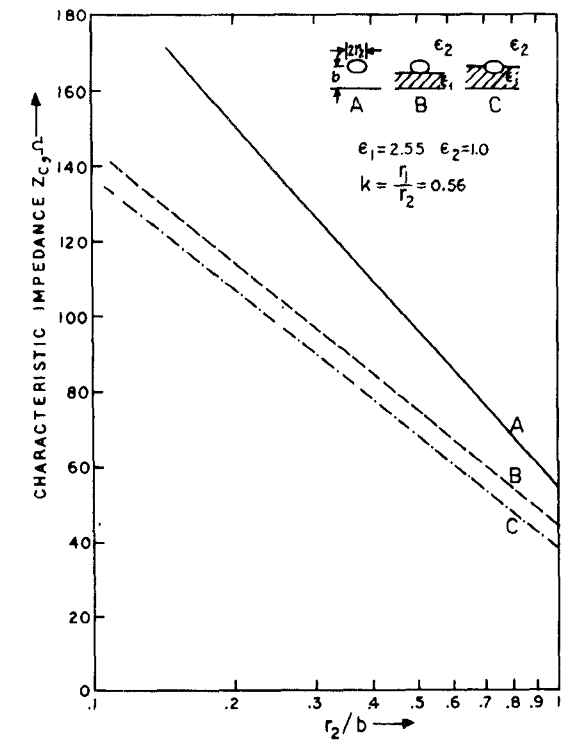


Fig. 6. The variation of characteristic impedance for the case of an ellipse with major axis above and parallel to the ground plane as a function of r_2/b with dielectric height as a parameter.

Using (10), (12)–(14), and [4, eqs. (16) and (17)] for the conformal transformation, the variation of the characteristic impedance with r_2/b for $\epsilon_1 = 2.55$ and $\epsilon_2 = 1$ is evaluated for the following cases.

Case 1: The ellipse is so placed that its major axis is parallel to the ground plane and

a) the dielectric interface is tangential to the ellipse at the lowermost point of the minor axis, and
b) the major axis coincides with the dielectric interface.

The results on characteristic impedance for the above cases with $k = r_1/r_2 = 0.56$ are presented in Fig. 6.

Case 2: *Circular Conductor above a Ground Plane*—The circle is so placed on the dielectric that

c) the lowermost point is on the dielectric interface, and
d) the diameter parallel to the ground plane coincides with the dielectric interface.

The results on the characteristic impedance for the above cases with $k = r_1/r_2 = 1$, 1) $\epsilon_1 = \epsilon_2 = 1$ and 2) $\epsilon_1 = 2.55$ and $\epsilon_2 = 1$ are presented in Fig. 7. The case A of Fig. 7 has a simple exact formula [2]. The numerical results for the case where the circle is exactly placed above the dielectric interface are compared with those of Wheeler [2].

Case 3: The ellipse is so placed that its minor axis is parallel to the ground plane and

e) the dielectric interface is tangential to the ellipse at the lowermost point of the major axis, and
f) the minor axis coincides with the dielectric interface.

The impedance data for the above cases with $k = r_1/r_2 = 1.9$ are presented in Fig. 8.

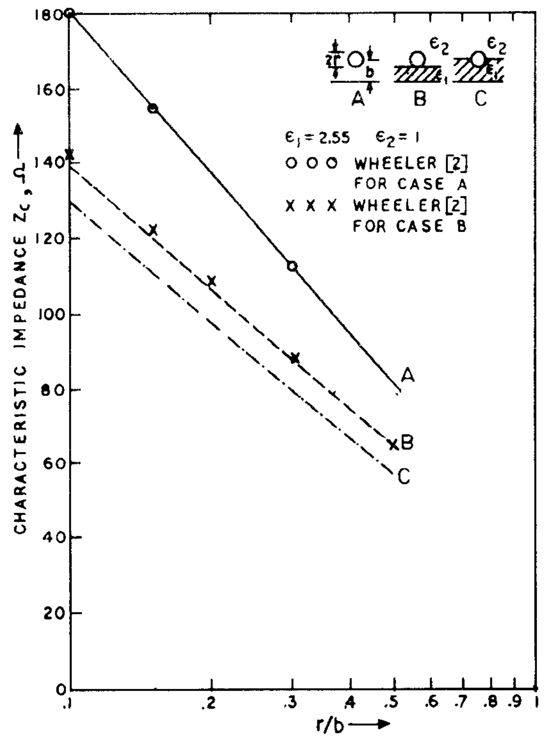


Fig. 7. The variation of the characteristic impedance for a circular conductor above a ground plane as a function of r/b with dielectric height as a parameter.

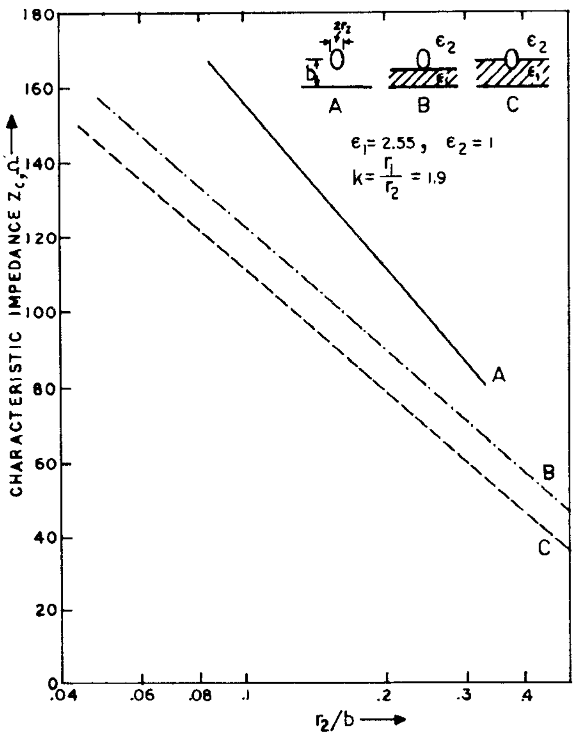


Fig. 8. The variation of the characteristic impedance for an ellipse with major axis above and perpendicular to the ground plane as a function of r_2/b with dielectric height as a parameter.

IV. Discussions

The analysis presented in this paper is based on the quasi-TEM-mode approximation. The results of the analysis are therefore valid for low frequencies only. Conformal transformation becomes ineffective to derive the characteristic impedance as the frequency increases. Using the formulation based on the hybrid mode analysis [7], the numerical results on the characteristic impedance have been evaluated for a symmetric stripline with substrate thickness equal to 1/16 in and stripwidth to spacing ratio of 1, the region between the strip and one of the ground planes being filled with dielectric having $\epsilon_r = 4$. The calculations based on the formulation and data supplied in the literature [7] reveal that the characteristic impedance changes from 63 Ω to approximately 59.5 Ω as frequency changes from 0 to 10 GHz.

The results of Fig. 2, 3, and 4 indicate that the characteristic impedance decreases monotonically with an increase in the major axis of the elliptic center conductor for a fixed eccentricity. Further, it is found that for an offset ratio of 0.2 and fixed r_1/r_2 ratio, the dielectric loading reduces the impedance by an amount ranging from 25 to 35 percent of its value for the corresponding air-filled case as the height of the dielectric is varied from the lower most point to the top most point of the elliptic center conductor. The results presented in Figs. 2, 3, and 4 also reveal that, as the orientation of the major axis of the ellipse is changed from a horizontal to a vertical position, the characteristic impedance of the line decreases. The fall in impedance for fixed eccentricity is sharper for the vertical orientation of the major axis. For the case of a circular conductor above a ground plane, the results obtained by the present method show an excellent agreement with those obtained by Wheeler [2] for $\epsilon_r = 1$. For $\epsilon_r = 2.55$, the maximum deviation between the results obtained by the present method from those of Wheeler [2] is less than 3 percent. This can be attributed to the fact that Wheeler assumed that the effective dielectric constant has a constant value of $\sqrt{\epsilon_r}$ for all values of r/b, which is equal to 1.59 when $\epsilon_r = 2.55$. From the results of the present analysis (Fig. 7), it is found that the effective dielectric constant $\epsilon_{r\text{eff}}$ varied from 1.68 to 1.58 as r/b is changed from 0.1 to 0.4.

It is worthwhile to mention that the present analysis for the case of an ellipse above a single ground plane reduces to a thin strip above a ground plane when r_1/r_2 ratio is equal to zero. This exactly corresponds to the case studied by Joshi *et al.* [3], the results of which are in excellent agreement with those of Wheeler [8].

Acknowledgment

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